Triple product p-adic L-functions

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Contents

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π^1 π^2 π^3 cuspidal automorphic representations $\mathrm{GL}_2(F)$ of weight and central character $(\underline{k_1}, \chi_1)$ $(\underline{k_2}, \chi_2)$ $(\underline{k_3}, \chi_3)$ respectivelly.
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IDEA: p-adically interpolate $L(\pi^1 \otimes \pi^2 \otimes \pi^3, \frac{1}{2})$, when $(\underline{k_1}, \chi_1)$ $(\underline{k_2}, \chi_2)$ $(\underline{k_3}, \chi_3)$ vary p-adically.

Ichino's formula

G multiplicative B/F, π^i_{JL} Jacquet-Langlands of π^i, $\phi^i \in \pi^i_{JL}$

$$\left(\int_{G(F)\mathbb{A}^\times\backslash G(\mathbb{A})} \phi^1(g)\phi^2(g)\phi^3(g)dg\right)^2 = L(\pi^1\otimes\pi^2\otimes\pi^3, \tfrac{1}{2})\prod_v \alpha_v(\phi^1_v,\phi^2_v,\phi^3_v).$$

idea: Interpolate $\left(\int_{G(F)\mathbb{A}^\times\backslash G(\mathbb{A})} \phi^1(g)\phi^2(g)\phi^3(g)dg\right)^2$.

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FACT:

$$\alpha_v(\pi^1_v, \pi^2_v, \pi^3_v) = 0 \Leftrightarrow \varepsilon_v(\pi^1 \otimes \pi^2 \otimes \pi^3) = -1 \Leftrightarrow \alpha_v(\pi^1_{JL,v}, \pi^2_{JL,v}, \pi^3_{JL,v}) \neq 0$$

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Our setting "indefinite", $\infty\setminus S = \tau$.

Modular forms

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Modular form weight $\underline{k}$;

$$\varphi \in \mathrm{Hom}_{G(\mathbb{R})}(D(k_\tau)\times \mathrm{Sym}^{k_\sigma-2}(\mathbb{C}^2),\mathcal{A})$$

where $\mathcal{A}$ is the space of automorphic forms of G.

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$f = \frac{\varphi(f_{k_\tau})}{f_{k_\tau}} : \mathfrak{H}\times G(\mathbb{A}_f)\to \mathrm{Sym}^{k_\sigma-2}(\mathbb{C}^2)^\vee$,

$$f(\gamma z,\gamma g) = (cz+d)^{k_\tau}\gamma f(z,g),$$

for $\gamma\in G(F)$

Local signs at places at ∞

$k_1, k_2, k_3 \in \mathbb{Z}$, $v \mid \infty$. Since $\chi_1\chi_2\chi_3 = 1$,

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- $\varepsilon_v(D(k_1), D(k_2), D(k_3)) = -1$ iff k_1, k_2, k_3 balanced

$$k_r < k_s + k_t, \qquad \forall r, s, t.$$

$$D(k_i)^{JL} = \mathrm{Sym}^{k_i - 2}(\mathbb{C}^2),$$

$$\alpha_v(\mathrm{Sym}^{k_1-2}(\mathbb{C}^2), \mathrm{Sym}^{k_2-2}(\mathbb{C}^2), \mathrm{Sym}^{k_3-2}(\mathbb{C}^2))?$$

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- $\varepsilon_v(D(k_1), D(k_2), D(k_3)) = 1$ iff k_1, k_2, k_3 unbalanced

$$k_1 \geq k_2 + k_3, \text{ or } k_2 \geq k_1 + k_3, \text{ or } k_3 \geq k_1 + k_2.$$

$$\alpha_v(D(k_1), D(k_2), D(k_3))?$$

Trilinear forms

- $\mathrm{Sym}^k(\mathbb{C}^2)^\vee \simeq \mathrm{Sym}^k(\mathbb{C}^2)$, $\mu \longmapsto P_\mu(X,Y) = \mu\left(\begin{vmatrix} X & Y \\ x & y \end{vmatrix}^k\right)$.

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  $\mu_1 \otimes \mu_2 \otimes \mu_3 \longmapsto \mu_1\mu_2\mu_3(\Delta(k_1,k_2,k_3))$

  $$\Delta(k_1,k_2,k_3) = \begin{vmatrix} X_2 & Y_2 \\ X_3 & Y_3 \end{vmatrix}^{\frac{k_2+k_3-k_1}{2}} \begin{vmatrix} X_1 & Y_1 \\ X_3 & Y_3 \end{vmatrix}^{\frac{k_1+k_3-k_2}{2}} \begin{vmatrix} X_2 & Y_2 \\ X_1 & Y_1 \end{vmatrix}^{\frac{k_2+k_1-k_3}{2}}$$

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  $$P_1 \otimes P_2 \longmapsto \sum_{n=0}^{k_3^*} \binom{k_3^*}{n} (-1)^n \frac{\partial^{k_3^*} P_1}{\partial X^{k_3^*-n} \partial Y^n} \frac{\partial^{k_3^*} P_2}{\partial X^n \partial Y^{k_3^*-n}}$$

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- $\alpha_v : D(k_1) \otimes D(k_2) \to D(k_3)$ $\quad \left(k_3^* = \frac{k_1+k_2-k_3}{2} \le 0\right)$

  $$f_1 \otimes f_2 \longmapsto \sum_{n=0}^{-k_3^*} (-1)^n \binom{-k_3^*}{n} \binom{N-2}{k_1+n-1} \delta_{k_1}^n(f_1)\delta_{k_2}^{-k_3^*-n}(f_2)$$

  δ_k *Shimura-Mass operator.*

Definite setting over $\mathbb{Q}$

Assume $F = \mathbb{Q}$, $\quad \varphi_i \in \mathrm{Hom}_{G(\mathbb{R})}(\mathrm{Sym}^{k_i}(\mathbb{C}^2), \mathcal{A})$,

$$\int_{G(F)\backslash G(\mathbb{A})} \varphi_1\varphi_2\varphi_3(\Delta(k_1, k_2, k_3))(g)dg = C \cdot L(\pi_1 \otimes \pi_2 \otimes \pi_3, \tfrac{1}{2})$$

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STRATEGY:

- Let $\mathbf{k} : \mathbb{Z}_p^* \to \mathcal{R}$ locally analytic character (universal?)

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- p-adic L-function

$$L_p(\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3) = \sum_{g\in G(F)\backslash G(\mathbb{A}_f)/U} c_g \mathbf{f}_1(g)\mathbf{f}_2(g)\mathbf{f}_3(g)(\Delta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3))$$

Indefinite setting

Modular forms as sections

Again $[F:\mathbb{Q}]=2$. Assume p inert in F; $\tau,\sigma:\mathcal{O}_p\hookrightarrow\mathbb{C}_p$.

Shimura curve X_U/F, $\quad X_U(\mathbb{C})=G(F)\backslash(\mathfrak{H}\times G(\mathbb{A}_f)/U)$

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"Universal" p-divisible group (Carayol)

$$G=(G_1\subset\cdots\subset G_n\subset\cdots),\qquad G_n=\left(X_{U(p^n)}\times(p^{-n}\mathcal{O}/\mathcal{O})^{\oplus 2}\right)/(U(p^n)/U)$$

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Define the following sheaves:

- ω sheaf of differentials of G.

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- ω sheaf of differentials of G.
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Modular forms as sections

Again $[F:\mathbb{Q}]=2$. Assume p inert in F; $\tau,\sigma:\mathcal{O}_p\hookrightarrow\mathbb{C}_p$.

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Modular forms of weight (k_τ,k_σ):

$$f:\mathfrak{H}\times G(\mathbb{A}_f)\longrightarrow \mathrm{Sym}^{k_\sigma}(\mathbb{C}^2)^\vee,\quad f(\gamma z,\gamma g)=(cz+d)^{k_\tau}\gamma f(z,g),$$

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Ichino's formula: $f_i \in \Gamma(X_U, \omega^{k_\tau^i} \otimes \mathrm{Sym}^{k_\sigma^i}\mathcal{H}_\sigma)$

$$\langle t(f_1, f_2), f_3 \rangle = C \cdot L(\pi^1 \otimes \pi^2 \otimes \pi^3, 1/2)$$

Strategy for the triple product p-adic L-function

$(\mathbf{k^i}_\tau, \mathbf{k^i}_\sigma) : \mathcal{O}_p^* \to \mathcal{R}$ loc. analytic characters (universal?) $i = 1, 2, 3$

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- $\mathbf{t} = (\mathbf{t}_\tau, \mathbf{t}_\sigma)$
$$L_p(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \langle \mathbf{t}(\mathbf{f}_1, \mathbf{f}_2), \mathbf{f}_3 \rangle$$

Triple product p-adic L-functions

Santiago Mollina

CRM

July 2017